PMM U.S.S.R.,Vol.44,pp.574-581 0021-8928/81/5 0574 $7.50/0

UDC 531.36

ON THE STABILITY OF A NONAUTONOMOUS HAMILTONIAN SYSTEM UNDER SECOND-ORDER RESONANCE*

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The problem of the stability of the equilibrium position of a nonautonomous 2π-periodic Hamiltonian system with two degrees of freedom, in a nonlinear setting, is examined in the case when the multipliers of the linearized system are equal and correspond to a combination-type parametric resonance. The cases of prime and nonprime elementary divisors of the linear system's characteristic matrix are studied. Stability in a finite approximation, and formal stability or instability of the equilibrium position are proved, depending on the coefficients of the Hamilton function. Computation formulas are quoted.

1. We consider a nonautonomous Hamiltonian system with two degrees of freedom

$$\frac{dq_k}{dt}=\frac{\partial H}{\partial p_k},\quad \frac{dp_k}{dt}=-\frac{\partial H}{\partial q_k}\quad (k=1,2) \tag{1.1}$$

Let the Hamilton function $H=H(q_k,p_k,t)$ be continuous and 2π-periodic in t; let it be analytic in q_k and p_k in a neighborhood of the origin of the phase space of q_k and p_k, the origin being an equilibrium position of system (1.1); consequently, it is representable as the Taylor series

$$H=H_2+\ldots+H_m+\ldots,\quad H_m=\sum_{\nu_1+\nu_2+\mu_1+\mu_2=m} h_{\nu_1\nu_2\mu_1\mu_2}(t)\, q_1^{\nu_1}q_2^{\nu_2}p_1^{\mu_1}p_2^{\mu_2},\quad h_{\nu_1\nu_2\mu_1\mu_2}(t+2\pi)=h_{\nu_1\nu_2\mu_1\mu_2}(t) \tag{1.2}$$

We write the linearized system of equations with Hamiltonian H_2

$$d\mathbf{x}/dt=\mathbf{Jh}(t)\,\mathbf{x},\quad \mathbf{x}=(q_1,q_2,p_1,p_2)^T,\quad \mathbf{J}=\left\|\begin{matrix}\mathbf{O}_2 & \mathbf{E}_2\\ -\mathbf{E}_2 & \mathbf{O}_2\end{matrix}\right\|,\quad \mathbf{h}(t)=\left\|\frac{\partial^2 H_2}{\partial \mathbf{x}^2}\right\|,\quad \mathbf{h}(t+2\pi)=\mathbf{h}(t) \tag{1.3}$$

where $\mathbf{O}_2$ and $\mathbf{E}_2$ are the null and the unit matrices of appropriate orders. By $\mathbf{X}(t)$ we denote the fundamental matrix of solutions of equation system (1.3), satisfying the initial conditions $\mathbf{X}(0)=\mathbf{E}_4$. As is well known /1—3/, the characteristic equation of system (1.3) $\det\|\mathbf{X}(2\pi)-\rho E_4\|=0$, is reflexive

$$\rho^4-a_1\rho^3+a_2\rho^2-a_1\rho+1=0 \tag{1.4}$$

(a_1 is the trace of matrix $\mathbf{X}(2\pi)$, a_2 is the sum of all its principal second-order minors) and together with the root ρ of Eq. (1.4) has the root $1/\rho$. Consequently /1—3/, for the stability of systems (1.3) and (1.1) it is necessary that $|\rho_j|=1$ $(j=1,\ldots,4)$. Henceforth we assume the fulfillment of these conditions. If among the roots of Eq. (1.4) there are no multiple ones, then system (1.3) is stable, but the stability of system (1.1) still does not follow from this.

The problem of the stability of the complete system in this case of unequal multipliers ρ_j has been solved in a number of papers (see /3/). Assertions on the Liapunov-instability or on formal stability have been obtained as functions of the coefficients of forms $H_2, H_3, H_4,\ldots$ (only the case of a simultaneous fulfillment of several resonance relations has remained unanalyzed). The case of equal multipliers is interesting from the theoretical viewpoint because the complete system (1.1) can be stable even if the linearized system (1.3) is unstable. In the majority of applied problems the case of multiple multipliers corresponds to the boundaries of the stability domain of the linear system and, therefore, the problem being studied here is closely related with the question on the "security" of the boundaries of the stability domain in parameter space /4/. We note another connection of the problem being examined with the analogous problem for autonomous systems. We number the roots of Eq. (1.4) in such a way that $\operatorname{Im}\rho_k\geqslant 0$, $\rho_{k+2}=\bar\rho_k$ (the overbar denotes the complex conjugate, and the necessary stability condition $|\rho_j|=1$ is fulfilled). The condition $|\rho_j|=1$ imposed on the multipliers is equivalent to all the characteristic indices $\pm i\lambda_k$ $(\rho_k=\exp(2\pi i\lambda_k))$ being pure imaginary. Then the multipliers of the nonautonomous system can be equal only in one of three cases: a) $\lambda_1=\pm\lambda_2 \pmod 1$ when $\rho_1=\rho_2\neq\pm1$; b) $2\lambda_1\neq 0$, $2\lambda_2=0 \pmod 1$ when $\rho_1\neq\pm1$, $\rho_2=\bar\rho_2=\pm1$; c) $2\lambda_1=0$, $2\lambda_2=0 \pmod 1$ when $\rho_1=\bar\rho_1=\pm1$, $\rho_2=\bar\rho_2=\pm1$. For autonomous systems in which the

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quantities λ_1 and λ_2 play the role of the frequencies of the linear system, case a) corresponds to the case of equal frequencies (second-order resonance) /5—7/, case b) to the case of zero frequency (first-order resonance) /8/, and case c) to the case of two zero frequencies (double first-order resonance).

The aim of the present paper is to solve the stability problem for the trivial equilibrium position of system (1.1) in the case when the multipliers of system (1.3) are equal and the characteristic indices satisfy the relation $\lambda_1 = \pm\lambda_2 \pmod 1$. In the notation adopted this signifies that $\rho_1 = \rho_2 = \rho \neq \pm 1$, i.e., the coefficient of Eq. (1.4) satisfy the relation $a_2 = 2 + a_1^2/4$ $(a_1 \neq \pm 4)$ and the numbers λ_1 and λ_2 are determined from them by the formulas

$$\lambda_1 = \pm\frac{1}{2\pi}\arccos\frac{a_1}{4} + k_1, \quad \lambda_2 = \pm\frac{1}{2\pi}\arccos\frac{a_1}{4} + k_2, \quad (a_1 = 4\operatorname{Re}\rho)$$

where k_1, k_2 are arbitrary integers. As will be seen from what follows, the answer to the stability question is independent of numbers k_1 and k_2. Therefore, we can take $\lambda_1 = \delta\lambda_2 = \lambda$, where $0 < \lambda < 1$, $\lambda \neq 1/2$, $\delta = \pm 1$. In actual mechanical problems the case being examined corresponds to the boundary of the domain of combined-type parametric resonance.

2. As for the autonomous systems, depending on the elements of matrix $\mathbf{X}(2\pi)$, it is necessary in the problem being examined to investigate separately the cases of prime and nonprime elementary divisors of the linearized system's characteristic matrix. At first we consider the case of nonprime elementary divisors. We note that for applications this case is more important than the case of prime elementary divisors, considered in Sect.3, since for the realization of the latter we need, besides the fulfillment of the resonance conditions, also the fulfillment of certain (equality-type) conditions imposed on the elements of matrix $\mathbf{X}(2\pi)$.

Let us normalize the linear system (1.3). According to Liapunov's reducibility theorem /1/ system (1.3) can be reduced, by means of a nonsingular linear substitution, to a system with constant coefficients. Many papers (see the bibliography in /2,3/) have dealt with solving the problem of normalizing linear canonic systems. Below we describe a constructive method for reducing system (1.3) to normal form, analogous to the method suggested by Markeev /3/ for the case of unequal multipliers.

Theorem 2.1. A real symplectic matrix $\mathbf{N}(t)$, continuously differentiable and 2π-periodic in t, exists such that the substitution

$$\mathbf{x} = \mathbf{N}(t)\mathbf{x}' \quad (\mathbf{x}' = (q_1', q_2', p_1', p_2')^T) \tag{2.1}$$

leads the quadratic part H_2 of the Hamiltonian of system (1.1) to the normal form $(\delta = \pm 1)$

$$H_2' = {}^1/_2\delta(q_1'^2 + q_2'^2) + \lambda(q_1'p_2' - q_2'p_1') \tag{2.2}$$

coinciding with the normal form of the autonomous problem /5,6/.

We prove the theorem by constructing the matrix $\mathbf{N}(t)$. We seek it in the form /3/

$$\mathbf{N}(t) = \mathbf{X}(t)\mathbf{A}e^{-\mathbf{B}t}\mathbf{C} \tag{2.3}$$

where

$$\mathbf{B} = \begin{Vmatrix} i\lambda & 1 & 0 & 0 \\ 0 & i\lambda & 0 & 0 \\ 0 & 0 & -i\lambda & 0 \\ 0 & 0 & -1 & -i\lambda \end{Vmatrix}, \quad \mathbf{C} = \frac{1}{\sqrt{2}}\begin{Vmatrix} 0 & 0 & -\delta & -i\delta \\ 1 & i & 0 & 0 \\ \delta & -i\delta & 0 & 0 \\ 0 & 0 & 1 & -i \end{Vmatrix} \tag{2.4}$$

and we select the constant matrix $\mathbf{A}$ in such a way that transformation (2.1) is real (i.e., $\mathbf{N}(t) = \overline{\mathbf{N}}(t)$), univalent, canonic and 2π-periodic in t. We note that transformation (2.1) leads system (1.3) to the form

$$\frac{d\mathbf{x}'}{dt} = \mathbf{J}\mathbf{h}'\mathbf{x}', \quad \mathbf{h}' = \left\| \frac{\partial^2 H_2'}{\partial \mathbf{x}'^2} \right\|$$

independently of the form of the nonsigular matrix $\mathbf{A}$. Since the matrices $\mathbf{X}(t)$, $e^{-\mathbf{B}t}$, $\mathbf{C}$ are symplectic, for transformation (2.1) to be canonic and univalent the matrix $\mathbf{A}$ too must be symplectic, i.e.

$$\mathbf{A}^T\mathbf{J}\mathbf{A} = \mathbf{J} \tag{2.5}$$

From the requirement that matrix $\mathbf{N}(t)$ be 2π-periodic in t follows the condition

$$\mathbf{X}(2\pi)\mathbf{A} = \mathbf{A}e^{2\pi\mathbf{B}}, \quad e^{2\pi\mathbf{B}} = \begin{Vmatrix} \rho & 2\pi\rho & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \bar\rho & 0 \\ 0 & 0 & -2\pi\bar\rho & \bar\rho \end{Vmatrix} \tag{2.6}$$

Consequently, in the case of nonprime elementary divisors being considered the matrix $e^{2\pi B}$ is the symplectic normal Jordan form of matrix $X(2\pi)$, while the eigenvectors and adjoined vectors of matrix $X(2\pi)$, normalized by condition (2.5), are the columns of the matrix A reducing matrix $X(2\pi)$ to normal form. Therefore, we set $A = LD$, where the nonsingular matrix L is some solution of Eqs. (2.6) and the matrix D deals with the satisfaction of condition (2.5) of norming the eigenvectors and the adjoined vectors. Let the columns l_j of matrix L satisfy the relations

$$X(2\pi)l_1 = \rho l_1, \quad X(2\pi)l_2 = \rho l_2 + 2\pi\rho l_1, \quad X(2\pi)l_3 = \bar\rho l_3 - 2\pi\bar\rho l_4, \quad X(2\pi)l_4 = \bar\rho l_4 \tag{2.7}$$

Then, having chosen the vectors l_j such that $l_3 = \delta\bar l_2$, $l_4 = -\delta\bar l_1$ and having set

$$D = \begin{Vmatrix} D' & O_2 \\ O_2 & D'' \end{Vmatrix}, \quad D' = \begin{Vmatrix} d_1 & d_2 \\ 0 & d_1 \end{Vmatrix}, \quad D'' = \begin{Vmatrix} \bar d_1 & 0 \\ -\bar d_2 & \bar d_1 \end{Vmatrix}$$

we find that the matrix satisfies Eq. (2.6) independently of d_1, d_2, l_1, l_2 (from (2.7)), and matrix $N(t)$ is real. Thus, to satisfy the last condition (2.5) it remains only to select d_1 and d_2 from the l_1 and l_2 from (2.7) for this, having rewritten condition (2.5) as $A^TJA = D^TL^TJLD = D^TFD = J$, we investigate at first the properties of the matrix $F = \| f_{jn} \|$, $f_{jn} = (l_j, Jl_n)$. Since $(U, JV) = -(JU, V)$ is valid for any four-vectors U and V, the matrix F is skew-symmetric. Further, as when $\rho_1 \neq \rho_2$, from Eqs. (2.7) and the choice of l_3 and l_4 it follows that $f_{12} = f_{34} = 0$, $f_{13} = \bar f_{24}$, $f_{23} = -\bar f_{23}$ (f_{23} is a pure imaginary number).

Let us show that $f_{14} = -\delta(l_1, J\bar l_1) = 0$. Let M_4 be the Euclidean space spanned by vectors l_j. We consider its three-dimensional subspace M_3 orthogonal to vector Jl_4. It is invariant relative to a linear transformation by matrix $X(2\pi)$. Indeed, if $g \in M_3$ (i.e., $(g, Jl_4) = 0$), then

$$(X(2\pi)g, Jl_4) = (X(2\pi)g, JX(2\pi)l_4)/\bar\rho = (g, Jl_4)/\bar\rho = 0$$

The two-dimensional linear subspace M_2 spanned by vectors l_3 and l_4 also is invariant and is contained in M_3 (matrix $X(2\pi)$ is nonsingular). Consequently /9/, M_3 necessarily contains one more eigenvector of matrix $X(2\pi)$, i.e., the vector l_1, but this signifies that $f_{14} = (l_1, Jl_4) = 0$. Hence we obtain as well that $f_{24} = \bar f_{24}$ (f_{24} is a real number).

Thus, we have established the form of matrix F, and, hence, of matrix A^TJA. Equating the elements of the latter matrix to the elements of matrix J, we obtain the norming relations

$$d_1 = \bar d_1 = |(l_1, J\bar l_2)|^{-1/2}, \quad d_2 = -\bar d_2 = -{}^1/_2\delta(l_2, J\bar l_2)d_1^3, \quad \delta = \operatorname{sign}(l_1, J\bar l_2)$$

Finally, in the system with Hamiltonian (2.2) we make one more canonic transformation $q_k' = q_k''$, $p_k' = \delta p_k''$ with valence δ. Gathering up the results obtained, for the normalizing matrix $N(t)$ now having valence δ and leading function H_2 to form (2.2) wherein $\delta = 1$, we finally obtain the expression

$$N(t) = \sqrt{2}X(t)\| r_2, -s_2, -r_1, s_1 \| \begin{Vmatrix} Q(t) & O_2 \\ tQ(t) & Q(t) \end{Vmatrix}, \quad Q(t) = \begin{Vmatrix} \cos\lambda t & \sin\lambda t \\ -\sin\lambda t & \cos\lambda t \end{Vmatrix} \tag{2.8}$$

where r_k and s_k are the real and imaginary parts of the vectors $A_1 = d_1l_1$, $A_2 = d_2l_1 + d_1l_2$ which are the first columns of matrix A. This completes the proof of Theorem 2.1.

We note that the linear system with Hamiltonian (2.2) is unstable since the general solution contains a growing term of the form $t\sin\lambda t$. However, as we shall subsequently see, from this there still does not follow the instability of the complete system.

Further in this section we reckon that in the system with Hamiltonian (1.2) the linear normalization (2.1) with matrix (2.8) has already been effected and that the Hamiltonian's quadratic part has the form (2.2) wherein $\delta = 1$. The notation for the variables is left as before (without primes).

By the Deprit—Hori method we now make a nonlinear normalization (*)

$$(q_k, p_k) \to (Q_k, P_k) \quad (k = 1, 2) \tag{2.9}$$

in the complete system, such that the new Hamiltonian function

$$K = K_2 + \ldots + K_m + \ldots \tag{2.10}$$

is of a simpler form. It is more convenient to make the nonlinear normalization in complex variables connected with the real variables by the formulas

$$(q_1^*, q_2^*, p_1^*, p_2^*)^T = C(q_1, q_2, p_1, p_2)^T \tag{2.11}$$

*) Markeev, A.P. and Sokol'skii, A.G., Some computational algorithms for normalizing Hamiltonian systems. Preprint Inst. Prikl. Mat. Akad. Nauk SSSR, No.31, 1976.

where matrix $\mathbf{C}$ has been defined in (2.4) and $\delta = 1$. In the complex variables (below, the asterisk denotes that the corresponding function has been written in complex variables) $H_2^* = i\lambda(q_1^*p_1^* + q_2^*p_2^*) + q_2^*p_1^*$, while the coefficients of forms H_m^* satisfy the relations

$$h^*_{\mu_2\mu_1\nu_2\nu_1} = (-1)^{\nu_1+\mu_2}\bar{h}^*_{\nu_1\nu_2\mu_1\mu_2} \tag{2.12}$$

Substitution (2.9) is close to being an identical substitution. Therefore, $K_2 = H_2$ ($K_2^* = H_2^*$). The coefficients of form K_m^* are related with those of the corresponding form S_m^* of the generating function of the Deprit—Hori method and with those of form G_m^*, uniquely determined by the already-known forms $K_2^*, \ldots, K^*_{m-1}$, $S_3^*, \ldots, S^*_{m-1}$, $H_2^*, \ldots, H^*_{m-1}$, H_m^*, by differential equations of the following type

$$\left(\frac{d}{dt} + ir_{\nu_1\nu_2\mu_1\mu_2}\right)s^*_{\nu_1\nu_2\mu_1\mu_2} + (\nu_1+1)s^*_{\nu_1-1,\,\nu_2-1,\,\mu_1,\,\mu_2} - (\mu_2+1)s^*_{\nu_1,\,\nu_2,\,\mu_1-1,\,\mu_2+1} = k^*_{\nu_1\nu_2\mu_1\mu_2} - g^*_{\nu_1\nu_2\mu_1\mu_2} \ (r_{\nu_1\nu_2\mu_1\mu_2} = \lambda(\nu_1+\nu_2-\mu_1-\mu_2),\ \nu_1+\nu_2+\mu_1+\mu_2 = m) \tag{2.13}$$

We can deal with the function $k^*_{\nu_1\nu_2\mu_1\mu_2}(t)$ in such a way that Hamiltonian (2.10) is maximally simplified and that only a 2π-periodic solution of Eq. (2.13) relative to $s^*_{\nu_1\nu_2\mu_1\mu_2}(t)$ exists. If the number $r_{\nu_1\nu_2\mu_1\mu_2}$ is not an integer or if ν_2 and μ_1 do not simultaneously equal zero, then we can set $k^*_{\nu_1\nu_2\mu_1\mu_2} = 0$. If $r_{\nu_1\nu_2\mu_1\mu_2} = N$ (an integer) and $\nu_2 = \mu_1 = 0$, then in K_m^* it is impossible to annihilate the term with coefficient $k^*_{\nu_1\nu_2\mu_1\mu_2}$, but we can set

$$k^*_{\nu_1\nu_2\mu_1\mu_2}(t) = \varkappa_{\nu_1\nu_2\mu_1\mu_2}\exp(-ir_{\nu_1\nu_2\mu_1\mu_2}t), \quad \varkappa_{\nu_1\nu_2\mu_1\mu_2} = \frac{1}{2\pi}\int_0^{2\pi} g^*_{\nu_1\nu_2\mu_1\mu_2}(t)\exp(ir_{\nu_1\nu_2\mu_1\mu_2}t)\,dt \tag{2.14}$$

Here the numbers (2.14) possess property (2.12). We note that the thus-chosen coefficients of the new Hamilton function are invariant relative to the substitution $\lambda \to \lambda +$ integer, and it is precisely because of this (as was noted in Sect.1) we can assume $0 < \lambda < 1$.

In order to eliminate the explicit dependence of the coefficients on time we make one more canonic transformation $(Q_k^*, P_k^*) \to (Q_k^{**}, P_k^{**})$ by using the generating function $T = (Q_1^*P_1^{**} + Q_2^*P_2^{**})\exp(-i\lambda t)$. Then, finally, in the complex variables the Hamiltonian takes the normal form (the notation has been left the same for the variables)

$$K^* = Q_2^*P_1^* + \sum \varkappa_{\nu_1\nu_2\mu_1\mu_2}Q_1^{*\nu_1}Q_2^{*\nu_2}P_1^{*\mu_1}P_2^{*\mu_2} + K^*_{m+1} + \ldots \tag{2.15}$$

Here the summation is taken over nonnegative indices $\nu_1, \nu_2, \mu_1, \mu_2$ such that $3 \leqslant \nu_1 + \nu_2 + \mu_1 + \mu_2 \leqslant m$, $\nu_2^2 + \mu_1^2 \neq 0$, and $r_{\nu_1\nu_2\mu_1\mu_2} = N$ (an integer).

Restricting, as in the autonomous problem /5,6/, the analysis to terms of upto forth order ($m = 3, 4$), we come to the necessity of considering three essentially different cases: 1) $3\lambda \neq N$, $4\lambda \neq N$; 2) $3\lambda = N$ (because of the condition $0 < \lambda < 1$ it is sufficient to consider only $N = 1, 2$); 3) $4\lambda = N$ (here $N = 1, 2, 3$). In the case 1) the normal form of the Hamiltonian in real variables takes the form

$$K = K^{(0)} + K^{(1)} \tag{2.16}$$

$$K^{(0)} = \frac{1}{2}(Q_1^2 + Q_2^2) + A(P_1^2 + P_2^2)^2, \quad K^{(1)} = (P_1^2 + P_2^2)[B(Q_1P_2 - Q_2P_1) + C(Q_1^2 + Q_2^2)] + K_5 + \ldots \tag{2.17}$$

and is analogous to the normal form in the autonomous problem /5/. Here the real coefficient A, needed subsequently, is expressed in terms of the coefficients of Hamiltonian (1.2) (written after making the linear normalization (2.1)) by the formula

$$A = \frac{1}{4}\varkappa_{2002}, \quad \varkappa_{2002} = \frac{1}{2\pi}\int_0^{2\pi} g^*_{2002}(t)\,dt \tag{2.18}$$

$$g^*_{2002} = h^*_{2002} + h^*_{2010}s^*_{1002} + 2h^*_{1011}s^*_{2001} + 3h^*_{0012}s^*_{3000} + h^*_{2001}s^*_{0102} + 2h^*_{1002}s^*_{1101} + 3h^*_{0003}s^*_{2100}$$

$$h^*_{2002} = \frac{1}{2}(3h_{0040} + h_{0022} + 3h_{0004})$$

$$s^*_{3000} = F(3\lambda, h^*_{3000}), \quad s^*_{2100} = F(3\lambda, h^*_{2100} + 3s^*_{3000})$$

$$s^*_{2001} = F(\lambda, h^*_{2001}), \quad s^*_{1101} = F(\lambda, h^*_{1101} + 2s^*_{2001})$$

$$s^*_{1002} = F(-\lambda, h^*_{1002}), \quad s^*_{0102} = F(-\lambda, h^*_{0102} + s^*_{1002})$$

$$F(r, f) = e^{-irt}\left[-I(t) + \frac{I(2\pi)}{1 - e^{2\pi ir}}\right], \quad I(\tau) = \int_0^{\tau} f(t)\,e^{irt}\,dt$$

$$h_{3000}^* = \frac{1}{2\sqrt{2}}[(-h_{0030} + h_{0012}) + i(h_{0021} - h_{0003})]$$

$$h_{2001}^* = \frac{1}{2\sqrt{2}}[(3h_{0030} + h_{0012}) + i(-h_{0021} - 3h_{0003})]$$

$$h_{2100}^* = \frac{1}{2\sqrt{2}}[(h_{1020} - h_{1002} - h_{0111}) + i(-h_{1011} - h_{0120} + h_{0102})]$$

$$h_{2010}^* = \frac{1}{2\sqrt{2}}[(h_{1020} - h_{1002} + h_{0111}) + i(-h_{1011} + h_{0120} - h_{0102})]$$

$$h_{1101}^* = \frac{1}{\sqrt{2}}[(-h_{1020} - h_{1002}) + i(h_{0120} + h_{0102})]$$

Theorem 2.2. If $A > 0$ in the normal form (2.16), (2.17), then the equilibrium position is formally stable. If $A < 0$, Liapunov-instability obtains.

To prove the theorem's first assertion we note that the above-described normalizing transformation (2.9) can be carried out in any order. Then the system admits of a formal (because of a possible divergence of transformation (2.9)) integral $K = \text{const}$ defined by formula (2.16) in which $K^{(1)}$ does not depend on time explicitly. Since function K is positive definite when $A > 0$, by the definition in /10/ the equilibrium position of the normalized (and, consequently, the original) system is formally stable. The instability is proved in the same way as in the autonomous problem /5/. We note that when $\varkappa_{2002} = 0$ we have $A = 0$ and the stability question is resolved by higher-order terms by considering the expressions $\varkappa_{3003}(P_1^2 + P_2^2)^3$, etc., in the normal form (2.16). The theorem has been proved.

Cases 2) and 3) are particular only in the nonautonomous problem. In case 2) the normal form in real variables takes form (2.16), where now

$$K^{(0)} = \frac{1}{2}(Q_1^2 + Q_2^2) + a(P_1^3 - 3P_1P_2^2) + b(3P_1^2P_2 - P_2^3), \quad K^{(1)} = K_4 + \cdots \tag{2.19}$$

$$a = -\frac{1}{\sqrt{2}}\operatorname{Re}\varkappa_{3000}, \quad b = -\frac{1}{\sqrt{2}}\operatorname{Im}\varkappa_{3000}, \quad \varkappa_{3000} = \frac{1}{2\pi}\int_0^{2\pi} h_{3000}^*(t)e^{3i\lambda_1 t}dt \tag{2.20}$$

Theorem 2.3. If $\varkappa_{3000} \neq 0$, the equilibrium position is unstable (the stability question when $\varkappa_{3000} = 0$ is resolved by Theorem 2.2).

The theorem is proved by constructing the Chetaev function /11/

$$V = -(Q_1P_1 + Q_2P_2)K^{(0)} \tag{2.21}$$

whose derivative relative to the equation with Hamiltonian (2.16), (2.19) is positive definite in the region $V > 0$ (an analogous function was first used by Chetaev when inverting the Lagrange–Dirichlet theorem).

In the case 3), in the normal form (2.16) now

$$K^{(0)} = {}^1/_2(Q_1^2 + Q_2^2) + K^{(0P)}(P_1, P_2)$$

$$K^{(1)} = (P_1^2 + P_2^2)[B(Q_1P_2 - Q_2P_1) + C(Q_1^2 + Q_2^2)] + K_5 + \ldots$$

$$K^{(0P)} = A(P_1^2 + P_2^2)^2 + a(P_1^4 - 6P_1^2P_2^2 + P_2^4) + 4b(P_1^3P_2 + P_1P_2^3)$$

$$a = {}^1/_2\operatorname{Re}\varkappa_{4000}, \quad b = {}^1/_2\operatorname{Im}\varkappa_{4000}, \quad \varkappa_{4000} = \frac{1}{2\pi}\int_0^{2\pi} g_{4000}^*(t)e^{4i\lambda_1 t}dt \tag{2.22}$$

$$g_{4000}^* = h_{4000}^* + 3h_{2010}^*s_{3000}^* + h_{2001}^*s_{2100}^*$$

$$h_{4000}^* = \frac{1}{4}[(h_{0040} - h_{0022} + h_{0004}) + i(-h_{0031} + h_{0013})]$$

Theorem 2.4. If the form $K^{(0P)}$ is positive definite in P_1 and P_2, the equilibrium position is formally stable. In the remaining case (excepting the case of sign-positiveness, when the stability question is resolved by higher-order terms) instability obtains.

The theorem's first assertion is proved in the same way as in Theorem 2.2. The instability is proved by using the Chetaev function (2.21).

3. Let us now consider the case of prime elementary divisors of the matrix $\mathbf{X}(2\pi) - \rho\mathbf{E}_4$ with $\rho_1 = \rho_2 \neq \pm 1$. At first (as in the preceding case) we describe briefly the linear normalizing procedure for system (1.3).

Theorem 3.1. A real symplectic matrix $\mathbf{N}(t)$ continuously differentiable and 2π-periodic in t exists such that the substitution (2.1) reduces the Hamiltonian H_2 of system (1.3) to normal form

$$H_2' = {}^1/_2\delta_1\lambda(q_1'^2 + p_1'^2) + {}^1/_2\delta_2\lambda(q_2'^2 + p_2'^2) \tag{3.1}$$

where the numbers $\delta_k = \pm 1$ are determined during the linear normalization.

We write the required matrix $N(t)$ in form (2.3), where now

$$\mathbf{B} = \operatorname{diag}\{i\lambda, i\lambda, -i\lambda, -i\lambda\},\quad \mathbf{C} = \frac{1}{\sqrt{2}}\left\|\begin{matrix}-\Delta & i\mathbf{E}_2\\ i\mathbf{E}_2 & -\Delta\end{matrix}\right\|,\quad \Delta = \left\|\begin{matrix}\delta_1 & 0\\ 0 & \delta_2\end{matrix}\right\| \tag{3.2}$$

and the arbitrary nonsingular matrix $\mathbf{A}$ satisfies the symplecticity condition (2.5) and the periodicity condition (2.6). We note that now the matrix $e^{2\pi\mathbf{B}}$ is the diagonal normal form of matrix $\mathbf{X}(2\pi)$, i.e., $\mathbf{A}$ is composed of the eigenvectors of matrix $\mathbf{X}(2\pi)$. Once again we set $\mathbf{A} = \mathbf{LD}$ and for the columns of matrix $\mathbf{L}$ we write relations analogous to (2.7) (the vectors $\mathbf{l}_j$ must be linearly independent),

$$\mathbf{X}(2\pi)\mathbf{l}_k = \rho\bar{\mathbf{l}}_k,\quad \mathbf{X}(2\pi)\mathbf{l}_{k+2} = \bar{\rho}\mathbf{l}_{k+2}\quad (k = 1, 2) \tag{3.3}$$

Then, having chosen $\mathbf{l}_{k+2} = i\delta_k\bar{\mathbf{l}}_k$ and having set

$$\mathbf{D} = \left\|\begin{matrix}\mathbf{D}' & \mathbf{O}_2\\ \mathbf{O}_2 & \mathbf{D}''\end{matrix}\right\|,\quad \mathbf{D}' = \left\|\begin{matrix}d_1 & d_2\\ 0 & d_3\end{matrix}\right\|,\quad \mathbf{D}'' = \left\|\begin{matrix}\bar{d}_1 & \delta_1\delta_2\bar{d}_2\\ 0 & \bar{d}_3\end{matrix}\right\|\quad (d_1 = \bar{d}_1,\ d_3 = \bar{d}_3)$$

(i.e., having satisfied the condition for $\mathbf{N}(t)$ to be real), we arrive at the problem of ascertaining the structure of a skew-symmetric matrix $\mathbf{F} = \mathbf{L}^T\mathbf{JL}$. Analysis shows that

$$\mathbf{F} = \left\|\begin{matrix}\mathbf{O}_2 & \mathbf{M}\\ -\mathbf{M} & \mathbf{O}_2\end{matrix}\right\|,\quad \mathbf{M} = \left\|\begin{matrix}f_{13} & f_{14}\\ \delta_1\delta_2\bar{f}_{14} & f_{24}\end{matrix}\right\|$$

where we can so choose the vector $\mathbf{l}_1$ (from the two linearly-independent vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ in (3.3)) that $f_{13} = i\delta_1(\mathbf{l}_1, \mathbf{J}\bar{\mathbf{l}}_1) \neq 0$. We write out the elements of matrix $\mathbf{A}^T\mathbf{JA}$ and we equate them to the elements of matrix $\mathbf{J}$ (the symplecticity condition). Solving the equations obtained relative to the elements of matrix $\mathbf{D}$, we obtain norming relations in which the signs of δ_1 and δ_2 in the coefficients of normal form (3.1) are selected such that the subradical expressions (they are real) are positive. Collecting the results obtained, we finally obtain for the normalizing matrix $\mathbf{N}(t)$ the expression

$$\mathbf{N}(t) = \sqrt{2}\,\mathbf{X}(t)\,\|\mathbf{r}_1, \mathbf{r}_2, \mathbf{s}_1, \mathbf{s}_2\|\left\|\begin{matrix}-\cos\lambda t\,\Delta & \sin\lambda t\,\mathbf{E}_2\\ -\sin\lambda t\,\Delta & -\cos\lambda t\,\mathbf{E}_2\end{matrix}\right\|$$

where $\mathbf{r}_k$ and $\mathbf{s}_k$ are the real and imaginary parts of the vectors $\mathbf{A}_1 = d_1\mathbf{l}_1$, $\mathbf{A}_2 = d_2\mathbf{l}_1 + d_3\mathbf{l}_2$ which are the first columns of matrix $\mathbf{A}$. This completes the proof of Theorem 3.1.

We note that now, in contrast to the case of nonprime elementary divisors considered in Sect.2, the linear system with Hamiltonian (3.1) is stable, although even here it does not follow that the complete system should be stable (see below). To carry out the nonlinear normalization we pass to the complex variables (2.11) wherein the matrix $\mathbf{C}$ is determined by (3.2). In complex variables we obtain $H_2^* = i\lambda(q_1^*p_1^* + q_2^*p_2^*)$, while the equation for determining the coefficients of the generating function of the Deprit—Hori method and the coefficients of the new Hamiltonian takes the form of relation (2.13) in which the last two summands on the left hand side are absent. Thus, analogously to formula (2.15) obtained for the case of nonprime elementary divisors, we now have

$$K^* = \sum \varkappa_{\nu_1\nu_2\mu_1\mu_2} Q_1^{*\nu_1} Q_2^{*\nu_2} P_1^{*\mu_1} P_2^{*\mu_2} + K_{m+1}^* + \ldots \tag{3.4}$$

where the summation is over indices $\nu_1, \nu_2, \mu_1, \mu_2$ such that $3 \leqslant \nu_1 + \nu_2 + \mu_1 + \mu_2 \leqslant m$, $r_{\nu_1\nu_2\mu_1\mu_2} = N$ (integer), and the coefficients $\varkappa_{\nu_1\nu_2\mu_1\mu_2}$ satisfy the relations

$$\varkappa_{\mu_1\mu_2\nu_1\nu_2} = i^{(\nu_1+\nu_2+\mu_1+\mu_2)}\delta_1^{(\nu_1+\mu_1)}\delta_2^{(\nu_2+\mu_2)}\bar{\varkappa}_{\nu_1\nu_2\mu_1\mu_2}$$

analogous to relations (2.12). Passing in (3.4) to real polar variables φ_k (coordinate) and r_k (momentum) by the formulas

$$Q_k^* = i\sqrt{r_k}\exp(i\delta_k\varphi_k),\quad P_k^* = -\delta_k\sqrt{r_k}\exp(-i\delta_k\varphi_k),\quad (k = 1, 2)$$

we obtain the final expression for the normal form of the Hamilton function

$$K = \sum i^{(\nu_1+\nu_2)}(-\delta_1)^{\mu_1}(-\delta_2)^{\mu_2}\varkappa_{\nu_1\nu_2\mu_1\mu_2}[r_1^{(\nu_1+\mu_1)}r_2^{(\nu_2+\mu_2)}]^{1/2} \times \exp\{i[\delta_1(\nu_1-\mu_1)\varphi_1 + \delta_2(\nu_2-\mu_2)\varphi_2]\} + K_{m+1} + \ldots \tag{3.5}$$

Theorem 3.2. Let $\delta_1\delta_2 > 0$. If $k\lambda \neq N$ (k, N are integers, $k = 3, \ldots, m$), then the equilibrium is stable when terms of upto order m, inclusive, are taken into account.

Carrying out the normalization of the Hamiltonian up to terms of order m, we can be convinced that the truncated system admits of a positive-definite integral $r_1 + r_2 = \text{const}$. This proves the theorem. We note, in addition, that if the number λ is irrational, then from this follows the formal stability of the equilibrium position.

Now let $\delta_1\delta_2 < 0$ and $3\lambda \neq N$, $4\lambda \neq N$. The Hamiltonian (3.5), normalized up to fourth-order terms (in q_k, p_k), takes a form analogous to the normal form in the autonomous problem /5/

$$K = K^{(0)} + K^{(1)}, \quad K^{(1)} = K_5 + \ldots \tag{3.6}$$

$$K^{(0)} = -a_{2020}r_1^2 - 2\delta_1\delta_2 r_1^{3/2} r_2^{1/2}(a_{2011}\cos\varphi - b_{2011}\sin\varphi) - r_1r_2(\delta_1\delta_2 a_{1111} + 2a_{2002}\cos 2\varphi - 2b_{2002}\sin 2\varphi) - \tag{3.7}$$

$$2\delta_1\delta_2 r_1^{1/2} r_2^{3/2}(a_{0211}\cos\varphi + b_{0211}\sin\varphi) - a_{0202}r_2^2, \quad \varphi = \delta_1\varphi_1 - \delta_2\varphi_2, \quad \varkappa_{\nu_1\nu_2\mu_1\mu_2} = a_{\nu_1\nu_2\mu_1\mu_2} + ib_{\nu_1\nu_2\mu_1\mu_2}$$

where the functions $K_5, \ldots$ are 2π-periodic in t and in the angular variables φ_1 and φ_2. Let us consider a function $\Phi(\varphi) = r^{-2}K^{(0)}|_{r_1=r_2=r}$ which is 2π-periodic in one variable $\varphi = \delta_1\varphi_1 - \delta_2\varphi_2$ (here $\delta_1\delta_2 < 0$).

Theorem 3.3. If form (3.7) is sign-definite in domain $r_1 \geqslant 0, r_2 \geqslant 0$ for any φ, then the equilibrium position is formally stable. If $\Phi(\varphi) \neq 0$ when $0 \leqslant \varphi < 2\pi$, but form (3.7) is not sign-definite, then the equilibrium position is stable when terms of upto fourth order in the expansion of the Hamiltonian function (1.2) are taken into account. If function $\Phi(\varphi)$ takes values of any sign (is sign-variable), then Liapunov-instability obtains.

We first prove the assertion on instability, assuming that a value φ^* exists such that $\Phi(\varphi^*) = 0$, but $\Phi'(\varphi^*) \neq 0$ (this restriction is unessential). Using the periodicity of $\Phi(\varphi)$, we choose a number ε such that the inequality $\Phi'(\varphi) < 0$ is fulfilled in the neighborhood $|\varphi - \varphi^*| < \varepsilon$. We consider the Chetaev function /3,5,8/

$$V = [r_2^\alpha - (r_1 - r_2)^2]\sin\Psi, \quad \Psi = \frac{\pi}{2\varepsilon}(\varphi_1 + \varphi_2 - \varphi^* + \varepsilon), \quad 2 < \alpha < 3 \tag{3.8}$$

As the region $V > 0$ we take the domain $r_2^\alpha - (r_1 - r_2)^2 > 0$, $|\varphi_1 + \varphi_2 - \varphi^*| < \varepsilon$. In this region, obviously, $r_1 = r_2 + \beta r_2^{\alpha/2}$, $|\beta| < 1$. For the derivative of function (3.8) by virtue of the equations of motion with Hamiltonian (3.6) we obtain

$$\frac{dV}{dt} = r_2^{\alpha+1}\left[\frac{\pi}{\varepsilon}(1 - \beta^2)\Phi(\varphi)\cos\Psi - \alpha\Phi'(\varphi)\sin\Psi\right] + o(r_2^{\alpha+1})$$

This function is positive definite in region $V > 0$ /5,8/, whence on the basis of Chetaev theorem /11/ we obtain the instability of the equilibrium position.

To prove the theorem's other assertion we note that the truncated system with Hamiltonian (3.7) has two integrals: $K^{(0)} = \text{const}$ and $r_1 - r_2 = \text{const}$, and, consequently, admits of the integral $G = (r_1 - r_2)^2 + [K^{(0)}]^2$ which is positive definite. Thus, on the basis of Liapunov theorem /1/ we obtain the stability of the complete system in the fourth order (if $k\lambda \neq N$, where $k = 3, \ldots, m$, then from this follows as well stability in the m-th order, while for an irrational λ, formal stability). Now let function (3.7) be sign-definite in r_1 and r_2 for any φ. In the system we carry the normalization out to terms of infinite order. This signifies that function (3.6) does not depend explicitly on time and, consequently, is a formal integral. Since this integral is sign-definite, the equilibrium position is formally stable according to the definition in /10/. We observe that this assertion of the theorem is valid also when $\delta_1\delta_2 > 0$. The theorem is proved.

The cases $3\lambda = N$ and $4\lambda = N$ for prime elementary divisors was not analyzed in detail. We note that this problem is analogous to the cases of simultaneous fulfillment of two resonance relations for multidimensional Hamiltonian systems the study of which is far from complete even in the simpler variants (see the survey /12/). Finally, all of the results described above carry over to the case of a nonautonomous system with $n + 2$ degrees of freedom if it is assumed that its frequencies $\lambda, \lambda_3, \ldots, \lambda_{n+2}$ are not connected by relations of parametric resonance.

The authors thank A.P. Markeev for attention to the work.

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Translated by N.H.C.